

Hadron multiplicity in e^+e^- events induced by top quark pairs at the ILC energy

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Abstract

The average charged hadron multiplicity in the e^+e^- events with the primary $t\bar{t}$ -pair at the collision energy 500 GeV, as well as the average multiplicity of charged hadrons from the top quark are calculated in QCD to be 86.7 ± 1.11 and 41.0 ± 0.54 , respectively.

1 Introduction

Experiments at LEP and SLAC revealed, besides other important results, quite interesting feature of the hadron multiple production dependent on the mass of the “primary” (anti)quarks which launch the process of the QCD evolution. It appeared that differences between the light and heavy quark-induced multiplicities become energy-independent. QCD calculations describe the phenomenon quite well.

Certainly, LEP could not give the information on the events induced by the top quarks. Recent discussions of the ILC project give us occasion to provide QCD predictions concerning the hadron multiple production in the events with primary t -quarks.

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We manage to calculate the average hadron multiplicity $N_{t\bar{t}}$ in the e^+e^- events with $t\bar{t}$ pair at the collision energy of the ILC with the following prediction:

$$N_{t\bar{t}}(W = 500 \text{ GeV}) = 86.67 \pm 0.55 . \quad (1)$$

We also theoretically calculated the average hadronic multiplicity from the top quark:

$$n_t = 41.03 \pm 0.27 . \quad (2)$$

Both values correspond to the average value of the top mass $m_t = 170.9$. Everywhere below, it is assumed that we deal with *average* multiplicities of *charged* hadrons.

The paper is organized as follows. In order to make our calculations of the hadron multiplicity in top quark events more easy for understanding, we consider first the multiple hadron production in $c\bar{c}$ ($b\bar{b}$) events. The hadron multiplicities in e^+e^- events associated with the $t\bar{t}$ -pair production are calculated in Section 3 in the framework of perturbative QCD. In Section 4 the numerical estimations and our main results are presented.

2 Hadron multiplicity in e^+e^- annihilation associated with $c\bar{c}$ or $b\bar{b}$ -pair production

Hadron multiplicity in $q\bar{q}$ event, $N_{q\bar{q}}(W)$, can be represented in the following general form [1]:

$$N_{q\bar{q}}(W) = 2n_q + C_F \int_{Q_0^2}^{W^2} \frac{dk^2}{k^2} \frac{\alpha_s(k^2)}{\pi} n_g(k^2) E_q\left(\frac{k^2}{W^2}\right) , \quad (3)$$

where q means a type of quarks produced in the process of e^+e^- annihilation into hadrons at the collision energy W . In what follows, the notation $q = Q$ (*heavy quark*) will mean charm or beauty quark, while the notation $q = l$ (*light quark*) will correspond to a massless case (when a pair of u , d or s -quarks is produced, whose masses are assumed to be equal to zero). The top quark production ($q = t$) will be studied in Sections 3 and 4.

The first term in the r.h.s. of Eq. (3), $2n_q$, is the multiplicity of primary (anti)quark of the type q (i.e. the multiplicity from the leading hadron which

contains this (anti)quark). It is taken from an analysis of the data ($2n_c = 5.2$, $2n_b = 11.1$ [3], and $2n_l = 2.4$ [4]).

The quantity $n_g(k^2)$ in (3) is the mean multiplicity of the gluon jet with a virtuality k^2 , for which we will take a QCD-based parametric form, with parameters fit to data, while $E_q(k^2/W^2)$ is the inclusive spectrum of the gluon jet emitted by primary quarks. It was explained in detail in Ref. [1] that one should not consider this mechanism of hadron production via gluon jets as due to “a single cascading gluon”. That quantity $E(k^2/W^2)$ is an inclusive spectrum of the gluon jets is seen, e.g., from the fact that the average number of jets $\int dk^2/k^2 E_q(k^2/W^2) \neq 1$.

Q_0 is a phenomenological parameter denoting the scale at which “preconfinement” of the off-shell partons occurs (as explained in Ref. [5]).

Let us introduce variables¹

$$\eta = \ln \frac{W^2}{k^2} \quad (4)$$

and

$$Y = \ln \frac{W^2}{Q_0^2} , \quad (5)$$

as well as notation

$$\hat{n}_g = \frac{C_F \alpha_s(k^2)}{\pi} n_g(k^2) , \quad (6)$$

where $C_F = (N_c^2 - 1)/2N_c$, and $N_c = 3$ is a number of colors. Then Eq. (3) can be represented as

$$N_{q\bar{q}}(Y) = 2 n_q + \int_0^Y d\eta \hat{n}_g(Y - \eta) E_q(\eta) \equiv 2 n_q + N_q(Y) . \quad (7)$$

In particular, $N_{l\bar{l}}(Y)$ means the multiplicity of hadrons in light quark events, while $N_{Q\bar{Q}}(Y)$ denotes the multiplicity of hadrons in a process when a pair of the heavy quarks is produced.

The physical meaning of the function

$$N_q(Y) = \int_0^Y d\eta \hat{n}_g(Y - \eta) E_q(\eta) \equiv \int_0^Y d\eta' \hat{n}_g(\eta') E_q(Y - \eta') \quad (8)$$

¹We will often use “rapidity-like” variables (analogous to η and Y) instead of the energy variable W throughout the paper.

is the following. It describes the average number of hadrons produced in virtual gluon jets emitted by the primary quark and antiquark of the type q . In other words, it is the multiplicity in $q\bar{q}$ event except for multiplicity of the decay products of the primary quarks at the final stage of hadronization (the terms $2n_q$ in (7)).

For the massless case, the function $E \equiv E_l$ was calculated in our paper [1]. In terms of variable

$$\sigma = \exp(-\eta) , \quad (9)$$

it looks as

$$\begin{aligned} E[\eta(\sigma)] = & (1 + 2\sigma + 2\sigma^2) \ln \frac{1}{\sigma} - \frac{3 + 7\sigma}{2} (1 - \sigma) - \sigma(1 + \sigma) \left(\ln \frac{1}{\sigma} \right)^2 \\ & + 4\sigma(1 + \sigma) \left[\frac{\pi^2}{12} + \ln \sigma \ln(1 + \sigma) + \text{Li}_2(-\sigma) \right] , \end{aligned} \quad (10)$$

where $\text{Li}_2(z)$ is the Euler dilogarithm. The function $E(\eta)$ is presented in Fig. 1. It has the asymptotics

$$E(\eta) \Big|_{\eta \rightarrow \infty} = \eta - \frac{3}{2} . \quad (11)$$

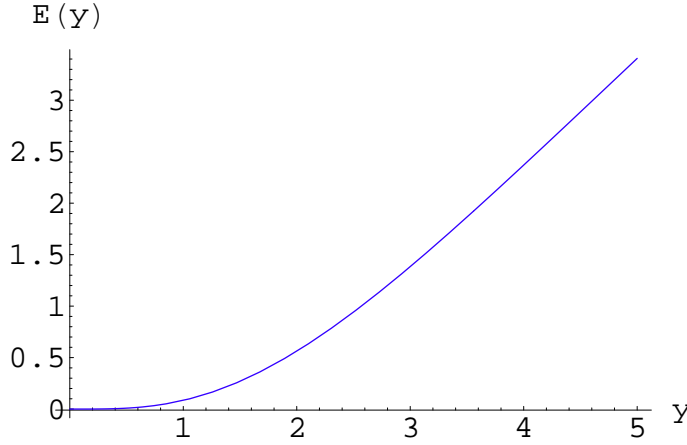


Figure 1: The function $E(y)$.

The derivative of $E(\eta)$ is positive, and $\partial E(\eta)/\partial \eta = 0$ at $\eta = 0$. As a result, the associated multiplicity $N_q(W)$ (8) is a monotonic increasing

function of the energy W for any positive function $n_g(k^2)$ since

$$\frac{\partial N_l(Y)}{\partial Y} = \int_0^Y d\eta \hat{n}_g(\eta) \frac{\partial E(Y - \eta)}{\partial Y} . \quad (12)$$

Now consider the multiplicity difference in events with the light and heavy flavors ($Q = c$ or b):

$$\delta_{Ql} = N_{Q\bar{Q}} - N_{l\bar{l}} . \quad (13)$$

At $W \gg m_Q$, one can neglect small power-like corrections $O(m^2/W^2)$. In such a case, the quantity δ_{Ql} is defined by [1]

$$\delta_{Ql} = 2(n_Q - n_l) - \Delta N_Q(Y_Q) , \quad (14)$$

where the notation

$$\Delta N_Q(Y_Q) = N_l - N_Q = \int_{-\infty}^{Y_Q} dy \hat{n}_g(Y_Q - y) \Delta E_Q(y) , \quad (15)$$

as well as variables

$$y = \ln \frac{m_Q^2}{k^2} \quad (16)$$

and

$$Y_Q = \ln \frac{m_Q^2}{Q_0^2} \quad (17)$$

are introduced. The lower limit of integration in Eq. (15), $-\ln(W^2/m_Q^2)$, is taken $-\infty$ because of the fast convergence of the integral at negative y .

Let us use another dimensionless variable

$$\rho = \exp(-y) . \quad (18)$$

The explicit form of ΔE_Q was derived in our paper [1] (see Fig. 2):

$$\begin{aligned} \Delta E_Q[y(\rho)] = & \left[1 + \rho \left(\frac{7}{2}\rho - 3 \right) \ln \frac{1}{\rho} + \left(\frac{9}{2} + 7\rho \right) \right. \\ & \left. + \rho (7\rho - 20) J(\rho) + 20 \frac{1 - J(\rho)}{\rho - 4} \right] . \end{aligned} \quad (19)$$

where

$$J(\rho) = \begin{cases} \sqrt{\frac{\rho}{\rho-4}} \ln \left(\frac{\sqrt{\rho} + \sqrt{\rho-4}}{2} \right), & \rho > 4, \\ 1, & \rho = 4, \\ \sqrt{\frac{\rho}{4-\rho}} \arctan \left(\frac{\sqrt{4-\rho}}{\rho} \right), & \rho < 4. \end{cases} \quad (20)$$

The function $\Delta E_Q(y)$ decreases at $y \rightarrow -\infty$ ($\rho \rightarrow \infty$) as

$$\Delta E_Q(y) \Big|_{y \rightarrow -\infty} \simeq \frac{11}{3} e^{-|y|}, \quad (21)$$

and has the following asymptotics at $y \rightarrow \infty$ ($\rho \rightarrow 0$):

$$\Delta E_Q(y) \Big|_{y \rightarrow \infty} \simeq y - \frac{1}{2}. \quad (22)$$

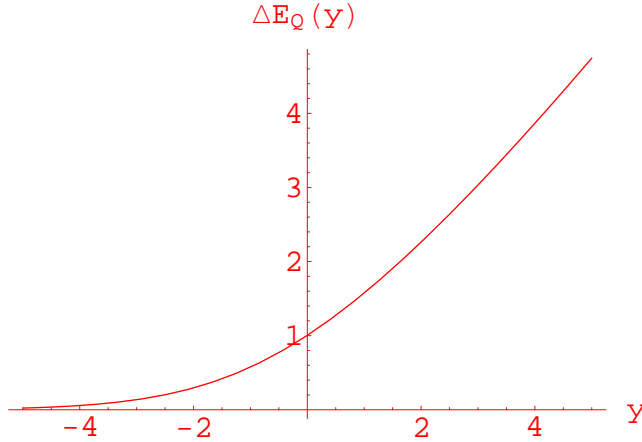


Figure 2: The function $\Delta E_Q(y)$.

Thus, we get the relation between average multiplicities of hadrons in $Q\bar{Q}$ and $l\bar{l}$ events:

$$N_{Q\bar{Q}}(W) = N_{l\bar{l}}(W) - \Delta N_Q(m_Q) + 2(n_Q - n_l), \quad (23)$$

with the multiplicity difference ΔN_Q defined by Eq. (15).

Our calculations [1] of the multiplicity differences $\delta_{Ql} = N_{l\bar{l}} - N_{Q\bar{Q}}$ ($Q = b, c$) with the use of formula (23) appeared to be in a good agreement with

the data. Recently we have reconsidered the QCD upper limit on quantity δ_{bl} [2] which appeared to be very close to all present experimental data on δ_{bl} .²

3 Hadron multiplicity in e^+e^- annihilation associated with $t\bar{t}$ -pair production

The goal of this paper is to calculate $N_{t\bar{t}}^h$, the average multiplicity of hadrons produced in e^+e^- events with the primary $t\bar{t}$ pair. We consider the case when the top (antitop) decay mode is pure hadronic. As a byproduct, we will calculate n_t , the hadron multiplicity of the on-shell top decay products.

We will assume that the square of the matrix element of the process $e^+e^- \rightarrow t^*\bar{t}^* \rightarrow X$ is factorized as follows:

$$\begin{aligned} |M(e^+e^- \rightarrow t^*\bar{t}^* \rightarrow \text{hadrons})|^2 &= |M(e^+e^- \rightarrow t^*\bar{t}^* \rightarrow t\bar{t} + \text{hadrons})|^2 \\ &\times |M(t \rightarrow \text{hadrons})|^2 \\ &\times |M(\bar{t} \rightarrow \text{hadrons})|^2, \end{aligned} \quad (24)$$

where $t^*(\bar{t}^*)$ denotes the virtual top quark (antiquark).

The factorization of the matrix element (24) means that there is no significant space-time overlap in the decay products of the on-shell t and \bar{t} -quarks.³ The QCD non-singlet evolution of the primary virtual t -quark is very slow because the difference of virtualities in logarithmic scale is very small down to the top quark mass. In other words, the virtual t -quark becomes “real” after just a few gluon radiation.

The effect of possible color reconnection was investigated by comparing hadronic multiplicities in $e^+e^- \rightarrow W^+W^- \rightarrow q\bar{q}'q\bar{q}'$ and $e^+e^- \rightarrow W^+W^- \rightarrow q\bar{q}'l\bar{\nu}_l$ events. No evidence for final state interactions was found by measuring the difference $\langle n_{4q}^h \rangle - 2\langle n_{2ql\bar{\nu}}^h \rangle$ [7, 8]. From the space-time point of view W bosons and t -quarks behave in a similar way, i.e. the latter manage to cover the distance $\Delta l \sim 1/\Gamma_t$, where Γ_t is the full width of the top. Since $\Gamma_t \simeq \Gamma_W$, we expect no interference effects in the decays of the on-shell t and \bar{t} -quarks.

²The data on N_{ll} as well as on δ_{bl} at different energies corrected for detector effects as well as for initial state radiation were recently cited in [6].

³Note that the off-shell t and \bar{t} -quarks fragment into hadrons through the emission of the gluon jets in a coherent way (the first term in the r.h.s of Eq. (24))

According to Eq. (24), the associative multiplicity in $t\bar{t}$ -event is given by the formula:

$$N_{t\bar{t}}^h(W, m_t) = N_t(W, m_t) + 2n_t, \quad (25)$$

where

$$N_t(W, m_t) = C_F \int_{Q_0^2}^{(W-2m_t)^2} \frac{dk^2}{k^2} \frac{\alpha_s(k^2)}{\pi} n_g(k^2) E_t(W^2, k^2, m_t^2). \quad (26)$$

Here and in what follows we will assume that the collision energy is a typical ILC energy, $W = 500$ GeV, for definiteness. In such a case, contrary to Eq. (8), power corrections $O(m_t/W)$ should be taken into account. The explicit form of the inclusive distribution of the gluon jets with the invariant mass $\sqrt{k^2}$ looks like

$$\begin{aligned} E_t(q^2, k^2, m^2) = & \left(-k \frac{\partial}{\partial k} \right) \int_1^A d\eta \left\{ \left[\frac{1}{\eta} \frac{(q^2 + k^2)^2 - 4m^4}{q^4} - 2 \frac{k}{q} \frac{q^2 + k^2 + 2m^2}{q^2} \right. \right. \\ & + 2\eta \frac{k^2}{q^2} \left. \right] \ln \left[\frac{\eta + \sqrt{\eta^2 - 1} \sqrt{(A - \eta)/(A_0 - \eta)}}{\eta - \sqrt{\eta^2 - 1} \sqrt{(A - \eta)/(A_0 - \eta)}} \right] \\ & - 2 \frac{k^2}{q^2} \sqrt{\eta^2 - 1} \frac{\sqrt{(A - \eta)}}{\sqrt{(A_0 - \eta)}} \\ & \times \left[1 + \frac{(1 + 2m^2/q^2)(1 + 2m^2/k^2)}{\eta^2 - (\eta^2 - 1)(A - \eta)/(A_0 - \eta)} \right] \left. \right\}, \quad (27) \end{aligned}$$

where $k \equiv \sqrt{k^2}$, $q \equiv \sqrt{q^2}$, and the following notations are introduced:

$$A = \frac{q^2 + k^2 - 4m^2}{2qk}, \quad A_0 = \frac{q^2 + k^2}{2qk}. \quad (28)$$

This formula has been derived by calculating QCD diagrams in the first order in the strong coupling constant.⁴ In the massless case ($m = 0$), we immediately come to the function $E(k^2/q^2)$ (10), while by neglecting small corrections $O(m^2/q^2)$, one can derive⁵ the explicit form of the function $\Delta E_Q(k^2/m^2) = E_Q(q^2, k^2, m^2) - E(q^2, k^2)$ (19). In our case ($q^2 = W^2$,

⁴See our comments after Eq. (3).

⁵After variables are properly changed.

$m = m_t$) we will estimate the integrals in Eqs. (26), (27) numerically (for details, see Section 4).

Now let us calculate another quantity in Eq. (25), n_t , which describes the hadronic multiplicity of the t -quark decay products. The top weakly decays into W^+ boson and b -quark. In its turn, the W^+ boson decays into a quark-antiquark pair.⁶ The quark-antiquark system results in massive jets which fragment into hadrons (see Fig. 3).

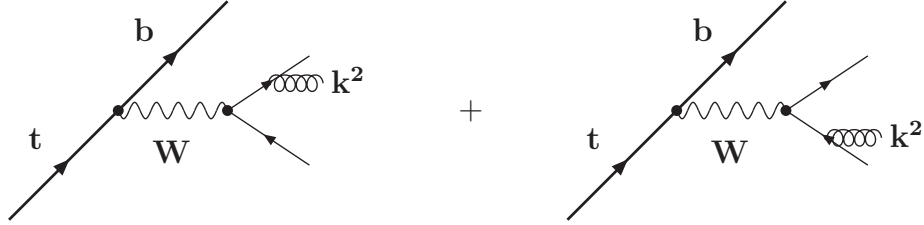


Figure 3: The emission of the gluon jets (spiral lines) by the quark pair resulting from the decay of the W^+ -boson in the first order in the strong coupling constant. The W^+ boson is produced in the weak decay of the top.

The gluon jets can be also emitted by the on-shell t -quark before its weak decay (the first diagram in Fig. 4) or by off-shell bottom quark (the second diagram in Fig. 4). At the end of these emissions, the on-shell b -quark

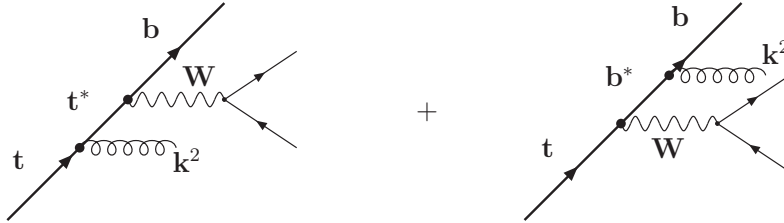


Figure 4: The emission of the gluon jets by the on-shell top quark and by off-shell bottom quark in the first order in the strong coupling constant. The emissions take place before and after the weak decay of the top, respectively. Off-shell quarks are denoted as t^* and b^* .

weakly decays into hadrons whose average multiplicity is equal to n_b . Since

⁶Remember that we are interested in hadronic decays of the W boson.

W -boson is a colorless particle, the diagrams in Fig. 4 do not interfere with those presented in Fig. 3.

Thus, the multiplicity n_t is a sum of three terms:

$$n_t = n_W + n_{tb} + n_b . \quad (29)$$

The quantity n_b is experimentally measurable one [3]. The first term in Eq. (29), n_W , is the hadron multiplicity of the W boson decay products. The second term, n_{tb} , is the hadron multiplicity in the gluon jets emitted by the on-shell top quark before its weak decay as well as by the bottom quark after the top decay.

3.1 Multiplicity of W boson decay products

The W^+ boson can decay either into two light quarks ($u\bar{d}$ and $u\bar{s}$ pairs) or into $c\bar{d}$ ($c\bar{s}$) pair. The former case is treated analogously to the light quark event in e^+e^- annihilation taken at the collision energy $W = m_W$. Here we will study the latter case.

Let $N_{Ql}(W)$ be hadronic multiplicity associated with the production of *one* heavy quark (antiquark) of the type Q and *one* light antiquark (quark) of the type l :

$$N_{Ql}(W) = (n_Q + n_l) + \hat{N}_{Ql}(W) . \quad (30)$$

Now let us introduce the notation (not to confuse with ΔN_Q from above):

$$\Delta N_{Ql} = N_l - \hat{N}_{Ql} . \quad (31)$$

Then the first term in the r.h.s. of Eq. (29) is given in terms of the function ΔN_{cl} by the formula:

$$n_W = N_{l\bar{l}}(Y_W) + \frac{1}{2} [-\Delta N_{cl}(Y_c) + n_c - n_l] , \quad (32)$$

where $Y_c = \ln(m_c^2/Q_0^2)$ and

$$Y_W = \ln \frac{m_W^2}{Q_0^2} . \quad (33)$$

The function $N_{l\bar{l}}(Y)$ in (32) is the hadronic multiplicity in light quark events.

Thus, we need to find an expression for ΔN_{cl} . Note that the formulae (15), (19) from Section 2 correspond to the case when *a pair* of heavy or *pair*

of light quarks is produced. Now we have to study the case when hadrons are produced in association with *a single* heavy quark (namely, *c*-quark) and one light quark.

Our QCD calculations result in the following representation for the multiplicity difference (see Appendix for details):

$$\Delta N_{Ql}(Y_Q) = \int_{-\infty}^{Y_Q} dy \hat{n}_g(Y_Q - y) \Delta E_{Ql}(y) , \quad (34)$$

with the dimensionless function ΔE_{Ql} :

$$\begin{aligned} \Delta E_{Ql}[y(\rho)] &= \frac{1}{4} [2 + \rho(3\rho - 2)] \ln \frac{1}{\rho} + \frac{1}{4} (5 + 6\rho) \\ &+ \frac{1}{2} \rho(3\rho - 8) J(\rho) + 6 \frac{1 - J(\rho)}{\rho - 4} . \end{aligned} \quad (35)$$

Here $\rho = \exp(-y)$. The quantity $J(\rho)$ was defined above (20). The function $\Delta E_{Ql}(y)$ is shown in Fig. 5.

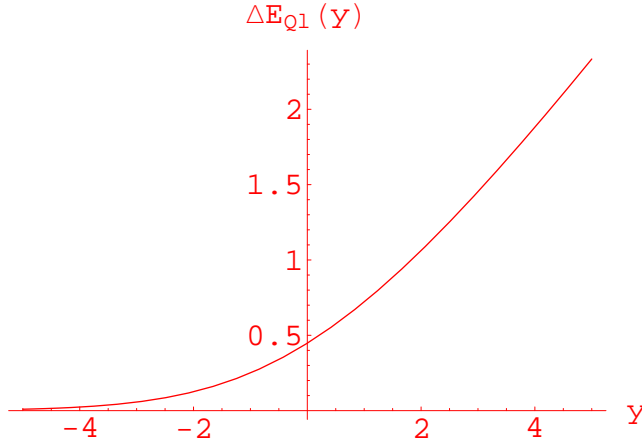


Figure 5: The function $\Delta E_{Ql}(y)$.

Since

$$\Delta E_{Ql}(y) \Big|_{y \rightarrow -\infty} \simeq \frac{3}{2} e^{-|y|} , \quad (36)$$

the integral (34) converges rapidly at the lower limit. Asymptotics of $\Delta E_{Ql}(y)$ at large y is the following:

$$\Delta E_{Ql}(y) \Big|_{y \rightarrow \infty} \simeq \frac{1}{2} \left(y - \frac{1}{2} \right) . \quad (37)$$

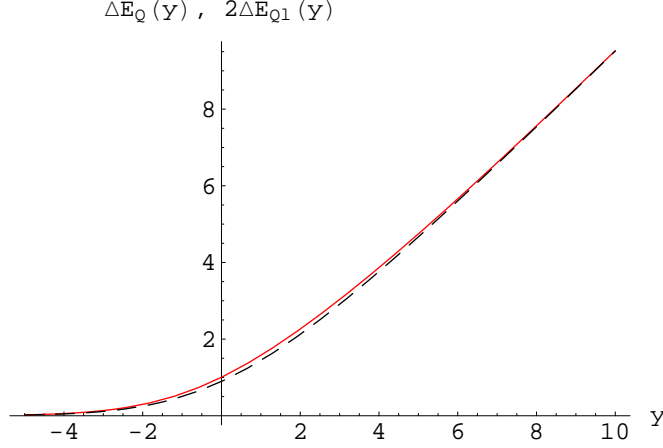


Figure 6: The function $\Delta E_Q(y)$ (solid line) vs. function $2\Delta E_{Ql}(y)$ (dashed line).

We derive from Eqs. (22), (37) that $\Delta E_{Ql}(y) = 0.5 \Delta E_Q(y)$ at large y . Numerical calculations show that $2\Delta E_{Ql}$ is very close to ΔE_Q at all y (see Fig. 6). Thus, we can put for our further numerical estimates:

$$\Delta N_{cl} = \frac{1}{2} \Delta N_c . \quad (38)$$

This relation means that

$$N_{Q\bar{l}} = \frac{1}{2} [N_{\bar{l}\bar{l}} + N_{Q\bar{Q}}] = N_{\bar{l}\bar{l}} + \frac{1}{2} \delta_{Ql} . \quad (39)$$

Correspondingly, we obtain:

$$n_W = N_{\bar{l}\bar{l}}(m_W) + \frac{1}{4} \delta_{cl} . \quad (40)$$

3.2 Multiplicity of top and bottom decay products

As was already said above, the on-shell top quark can emit jets before it weakly decays into W^+b . After the weak decay of the top, the off-shell b -quark “throws off” its virtuality by emitting massive gluon jets. The fragmentation of these massive gluon jets into hadrons results in the average hadron multiplicity n_{tb} .

To calculate the multiplicity n_{tb} , one has to derive the inclusive spectrum of the gluon jets, emitted by the top and bottom quarks. Let us denote it as E_{tb} . Then the multiplicity n_{tb} will be given by the formula:

$$n_{tb} = \int_0^{Y_{tb}} dy \hat{n}_g(Y_{tb} - y) E_{tb}(y) , \quad (41)$$

where

$$y = \ln \frac{(m_t - m_W - m_b)^2}{k^2} , \quad (42)$$

and

$$Y_{tb} = \ln \frac{(m_t - m_W - m_b)^2}{Q_0^2} , \quad (43)$$

with k^2 being the gluon jet invariant mass, $(m_t - m_W - m_b)^2$ its upper bound.

In the lowest order in the strong coupling constant, the quantity $E_{tb}(y)$ is given by two diagrams in Fig. 4. It is presented by an integral which depends on the ratio k^2/m_t^2 , as well as on mass ratios m_W^2/m_t^2 and m_b^2/m_t^2 . This integral cannot be calculated analytically, but can be estimated numerically. The function $E_{tb}(y)$ is presented in Fig. 7. It is worth to note that in the Feynman gauge the dominating contribution to $E_{tb}(y)$ comes from the *interference* of two diagrams shown in Fig. 4.

3.3 Associated multiplicity of hadrons in $t\bar{t}$ events

The formulae of the previous subsections enable us to derive the average multiplicity of the charged hadrons in e^+e^- annihilation at the collision energy W associated with the production of the $t\bar{t}$ -pair. It is of the form:

$$\begin{aligned} N_{t\bar{t}}^h(W, m_t) &= N_t(W, m_t) + 2[N_{l\bar{l}}(m_W) + n_{tb} + n_b] \\ &+ [-\Delta N_{cl}(m_c) + n_c - n_l] . \end{aligned} \quad (44)$$

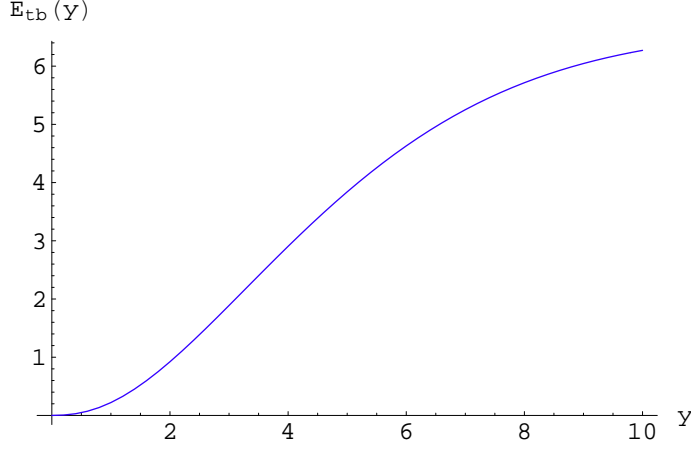


Figure 7: The function $E_{tb}(y)$.

Let us remind to the reader the meaning of all quantities in Eq. (44). The function $N_t(W, m_t)$ describes the average number of hadrons produced in association with the $t\bar{t}$ -primary pair, *except for* the decay products of the top(antitop) (26). The quantity $N_{l\bar{l}}(m_W)$ is the mean hadron multiplicity in the light quark event taken at the energy $E = m_W$. The hadron multiplicity n_{tb} comes from the emission by t and b quarks (41). The quantity n_l is the mean multiplicity of hadrons produced in the decay of the on-shell primary quark q ($q = l, c, b$). Finally, the combination $[\Delta N_{cl}(m_c) + n_l - n_c]$ is the difference of multiplicities in the processes with the primary $l\bar{l}$ - and $c\bar{c}$ -pairs. As for the hadron multiplicity resulting from the decay of the on-shell top (anti)quark, it is given by

$$\begin{aligned} n_t &= n_W + n_{tb} + n_b \\ &= N_{l\bar{l}}(m_W) + n_{tb} + n_b + \frac{1}{2} [-\Delta N_{cl}(m_c) + n_c - n_l] . \end{aligned} \quad (45)$$

The expressions for $N_{l\bar{l}}$, ΔN_{cl} are given by Eqs. (7), (34), respectively.

Let us stress that $N_{l\bar{l}}$ and n_q ($q = b, c, l$) are extracted from the data, and ΔN_{cl} can be related with the measurable quantities (see formulae (38)-(40) in the end of subsection 3.1):

$$\Delta N_{cl} \simeq \frac{1}{2} \Delta N_c = n_c - n_l - \frac{1}{2} \delta_{cl} . \quad (46)$$

Then we obtain:

$$n_t = N_{\bar{l}l}(m_W) + \frac{1}{4} \delta_{cl} + n_{tb} + n_b , \quad (47)$$

and

$$N_{t\bar{t}}^h(W, m_t) = N_t(W, m_t) + 2 \left[N_{\bar{l}l}(m_W) + \frac{1}{4} \delta_{cl} + n_{tb} + n_b \right] , \quad (48)$$

where n_{tb} is defined above (41). In what follows, we will use the value

$$\delta_{cl} = 1.03 \pm 0.34 \quad (49)$$

from Ref. [6].

4 Numerical estimates of hadron multiplicities.

In order to estimate the multiplicity of the decay products of the top (formula (47)), one has to know the energy dependence of the hadron multiplicity in the light quark event. The latter is defined by Eq. (7), where the function \hat{n}_g is related with gluon jet multiplicity $n_g(k^2)$ (6). We have fitted the data on $N_{\bar{l}l}(W)$ by using the following QCD-motivated expression for $n_g(k^2)$:

$$n_g(k^2) = a + b \exp \left[c \sqrt{\ln(k^2/Q_0^2)} \right] , \quad (50)$$

where $c = 1.63$, and k^2 is the invariant mass of the jet. We have got the following values of the parameters:

$$a = 3.89, \quad b = 0.01, \quad Q_0 = 0.87 \text{ GeV} . \quad (51)$$

The result of our fit is presented in Fig. 8 in comparison with the data. Note that $\chi^2/d.o.f.$ becomes twice smaller if one eliminates the experimental point at $W = 58$ GeV (open circle in Fig. 8), which lies much lower than neighboring points, is crossed out from the fit. In such a case, the values of the parameters are practically the same as in (51) with $\chi^2/d.o.f. = 0.90$.

For our numerical estimates we shall use $m_W = 80.40 \pm 0.03$ GeV [10] and the recent value of the top mass [9]:

$$m_t = 170.9 \pm 1.8 \text{ GeV} . \quad (52)$$

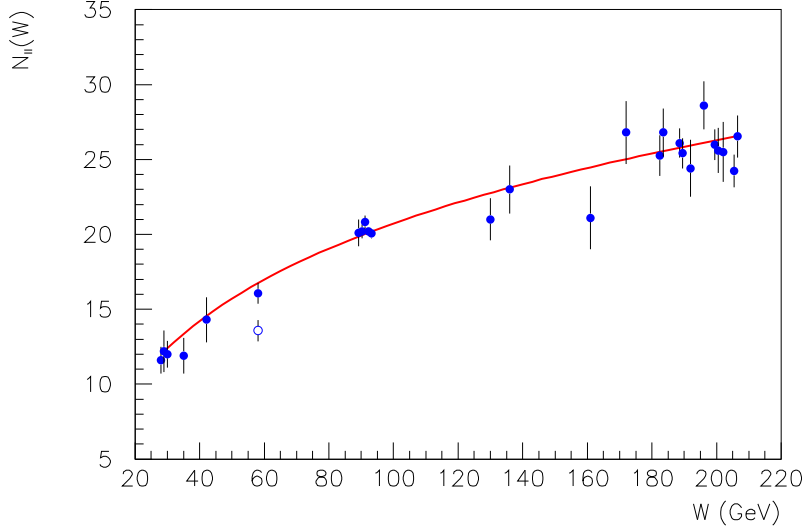


Figure 8: The fit of the data on the hadron multiplicity in light quark events (solid line, see formula in the text). The data are taken from Ref. [6].

As for the bottom quark, its pole mass is quoted in [10] to be $m_b = 4.7 - 5.0$ GeV. We will take the average value

$$m_b = 4.85 \pm 0.15 \text{ GeV} . \quad (53)$$

By using our fit, we obtain $N_{ll}(m_W) = 19.09$. Then we get (see Eqs. (40), (49), (41)):

$$n_W = 19.34 \pm 0.10 , \quad (54)$$

$$n_{tb} = 16.14 \pm 0.24 . \quad (55)$$

The error in Eq. (54) is defined by that of the multiplicity difference δ_{cl} (49), while that in Eq. (55) comes from uncertainties of the quark masses m_t (52) and m_b (53).

Our result (54) is in a nice agreement with the experimental values from Ref. [7],

$$n_W = 19.3 \pm 0.3 \pm 0.3 , \quad (56)$$

and Ref. [8],

$$n_W = 19.44 \pm 0.13 \pm 0.12 . \quad (57)$$

Now let us calculate the associated hadron multiplicity in $t\bar{t}$ event (48) at fixed energy $W = 500$ GeV. To do this, we need to estimate the multiplicity $N_t(W, m_t)$ by using formulae (26) and (27):

$$N_t(W = 500 \text{ GeV}) = 4.61 \pm 0.11 . \quad (58)$$

The errors in (58) come from top quark mass errors. It follows From Eqs. (61), (58) that $N_{t\bar{t}}(e^+e^- \rightarrow t\bar{t} \rightarrow \text{hadrons}) = 86.67 \pm 0.55$.

In order to estimate possible theoretical uncertainties, we have repeated our calculations taking the different form of the average multiplicity of the gluon jet (compare with the QCD-based expression (50)):

$$\hat{n}_g(k^2) = A + B \ln^2 \frac{k^2}{Q_0^2} . \quad (59)$$

It appeared that the data on the average multiplicity in light quark events can be fitted well by using this expression (with $A = 4.21$, $B = 0.012$ and $Q_0 = 0.93$ GeV). In particular, we have obtained the following average values for the hadronic multiplicities: $n_W = 19.52$, $n_{tb} = 16.43$, $N_t = 4.59$. Thus, theoretical uncertainties can be estimated to be 0.47 and 0.96 for n_t and $N_{t\bar{t}}$, respectively.

Taking into account the phenomenological value of n_b [6],

$$n_b = 5.55 \pm 0.09 , \quad (60)$$

we obtain from (47):

$$n_t(t \rightarrow \text{hadrons}) = 41.03 \pm 0.54 . \quad (61)$$

In the case when the W boson decays into leptons, we predict:

$$n_t(t \rightarrow l\bar{\nu}_l + \text{hadrons}) = 21.69 \pm 0.53 . \quad (62)$$

As a result, we obtain the average hadron multiplicity in e^+e^- annihilation with the primary $t\bar{t}$ -pair:

$$N_{t\bar{t}}(e^+e^- \rightarrow t\bar{t} \rightarrow \text{hadrons}) = 86.67 \pm 1.11 . \quad (63)$$

In the case when both W bosons decay into leptons, the mean multiplicity of the hadrons is expected to be

$$N_{t\bar{t}}(e^+e^- \rightarrow t\bar{t} \rightarrow W^+W^- + \text{hadrons}) = 47.99 \pm 0.59 . \quad (64)$$

Finally, we predict that

$$N_{t\bar{t}}(e^+e^- \rightarrow t\bar{t} \rightarrow b\bar{b} W^+W^- + \text{hadrons}) = 36.89 \pm 0.56 . \quad (65)$$

All estimations (63)-(65) correspond to the collision energy $W = 500$ GeV. We can mention the estimation of the hadron multiplicity from Ref. [11], $N_{t\bar{t}}(e^+e^- \rightarrow t\bar{t} \rightarrow b\bar{b} W^+W^- + \text{hadrons}) \approx 29$, which was obtained for $W = 390$ GeV and $m_t = 175$ GeV. For the same values of W and m_t , our formulae give $N_{t\bar{t}}(e^+e^- \rightarrow t\bar{t} \rightarrow b\bar{b} W^+W^- + \text{hadrons}) = 34.3$.

The formulae (61)-(65) is our main result. We hope that the hadron multiplicities of the top decay products (Eqs. (61) and (62)) will be measured at the LHC.

Acknowledgements

We are thankful to the referee for his comments and critical remarks that helped us to improve the presentation of some our results.

Appendix A

Here we present some formulae for the case when the W boson decays into hadrons via production of $c\bar{s}$ (or $c\bar{d}$) pair. Since the total width of the W boson, Γ_W , is much less than its mass, and the hadron multiplicity is a smooth function of energy, we will use zero width approximation and take the multiplicity at $W = m_W$. It can be shown that the account of the W boson width results in corrections which are numerically small (less than 1.6%, see Appendix B).

To calculate the inclusive spectrum of the gluon jets emitted by the decay products of the W -boson, we need to calculate two sub-diagrams of the diagram presented in Fig. 9.

The exterior part of the diagram in Fig. 9 describes the emission of the W boson by the top quark, with l being a 4-momentum of the t -quark, q is a 4-momentum of the W boson, and $(l - q)$ is a 4-momentum of the b -quark. The corresponding expression for this part of the diagram (after convolution

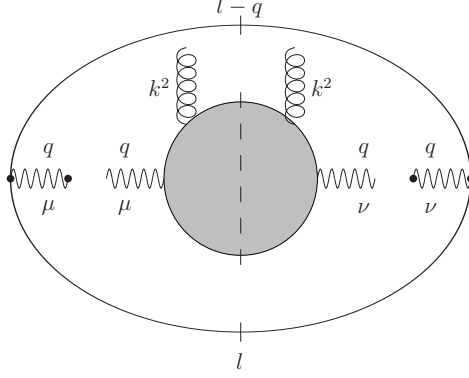


Figure 9: The generalized diagram describing the inclusive spectrum of the gluon jets (curly line) with the virtuality k^2 inside the W boson (wavy line). In its turn, the W boson is a product of the weak decay of the top quark (solid line with 4-momentum l).

with the tensor parts of the W boson propagators) looks like

$$\begin{aligned} \Pi_{\mu\nu}(l, q) = 4 \Big\{ & 4 l_\mu l_\nu - 2 (l_\mu q_\nu + q_\mu l_\nu) - g_{\mu\nu} (m_t^2 + m_b^2 - m_W^2) \\ & - \frac{2}{m_W^2} [(m_t^2 - m_b^2) l_\mu - m_t^2 q_\mu] q_\nu - \frac{2}{m_W^2} q_\mu [(m_t^2 - m_b^2) l_\nu - m_t^2 q_\nu] \\ & + \frac{1}{m_W^4} q_\mu q_\nu [(m_t^2 - m_b^2)^2 - q^2 (m_t^2 + m_b^2)] + 2i \varepsilon_{\mu\nu\rho\sigma} l^\rho q^\sigma \Big\}, \end{aligned} \quad (\text{A.1})$$

where m_t and m_b are masses of the top and beauty quark, respectively. In what follows, we will neglect power corrections of the type $\mathcal{O}(m_c/m_t)$ and $\mathcal{O}(m_b/m_t)$.

The inner part of the diagram in Fig. 9 describes the distribution of the massive gluon jet with the invariant mass k^2 produced by the W boson. Let $D^{\mu\nu}$ be the expression corresponding to this diagram. In the first order in the strong coupling constant, $D^{\mu\nu}$ is represented by the sum of three QCD diagrams presented in Figs. 10a, 10b and 10c.

Since $q^\mu \Pi_{\mu\nu} = 0$, one needs to calculate only two tensor structures in $D_{\mu\nu}$, namely, $g_{\mu\nu}$ and $k_\mu k_\nu$. The convolution of the tensor $\Pi_{\mu\nu}$ with the tensor $D^{\mu\nu} = g^{\mu\nu} D_1 + k^\mu k^\nu D_2 + \dots$,

$$A = \Pi_{\mu\nu} D^{\mu\nu} = g^{\mu\nu} \Pi_{\mu\nu} D_1(k^2, qk) + k^\mu k^\nu \Pi_{\mu\nu} D_2(k^2, qk), \quad (\text{A.2})$$

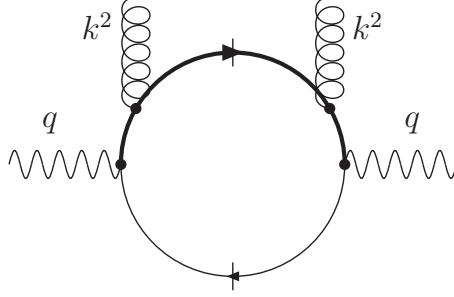


Figure 10a: The inclusive distribution of the massive gluon jet with the virtuality k^2 . The wavy line is the W boson, whose 4-momentum is q . The thick quark line is a heavy quark, while the thin line is a light quark. The cut quark lines mean that these quarks are on-shell quarks.

depends on Lorentz-invariant variables k^2, qk, lk . Moreover, it is a polynomial of the second order in variable lk . One can use the following useful relation:

$$\int \frac{d^4k}{(2\pi)^3} A(k^2, qk, lk) = \frac{1}{(2\pi)^2 m_t^2 (1-r)} \iiint dk^2 d(qk) d(lk) A(k^2, qk, lk) , \quad (\text{A.3})$$

where

$$r = \frac{m_W^2}{m_t^2} . \quad (\text{A.4})$$

It is naturally to integrate the function $A(k^2, qk, lk)$ first in variable lk , whose lower and upper limits are

$$(lk)_\pm = \frac{1}{2r} \left[(qk)(1+r) \pm (1-r) \sqrt{(qk)^2 - m_W^2 k^2} \right] , \quad (\text{A.5})$$

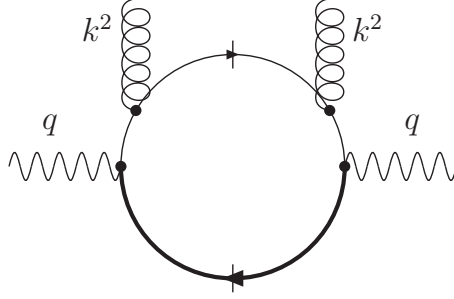


Figure 10b: The same as in Figs. 10a, but with the gluon jet emitted by the light quark.

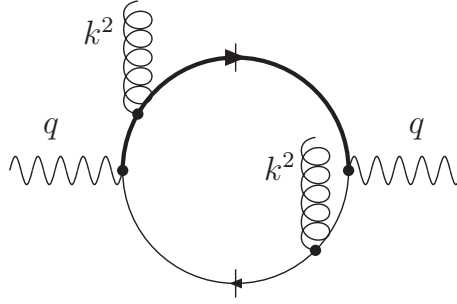


Figure 10c: The interference diagram which also contributes to the inclusive distribution of the gluon jets with the virtuality k^2 inside the W boson. The diagrams are taken in the sum of the diagrams with the factor 2.

by using the following formulae:

$$\begin{aligned}
 \int_{(lk)_-}^{(lk)_+} d(lk) &= \frac{1}{r} (1-r) \sqrt{(qk)^2 - m_W^2 k^2} , \\
 \int_{(lk)_-}^{(lk)_+} (lk) d(lk) &= \frac{1}{2r^2} (1-r^2) (qk) \sqrt{(qk)^2 - m_W^2 k^2} , \\
 \int_{(lk)_-}^{(lk)_+} (lk)^2 d(lk) &= \frac{1}{12r^3} (1-r) \left[4(qk)^2 (1+r+r^2) - m_W^2 k^2 (1-r)^2 \right] \\
 &\quad \times \sqrt{(qk)^2 - m_W^2 k^2} .
 \end{aligned} \tag{A.6}$$

Note that the parts of tensors $\Pi_{\mu\nu}$ and $D^{\mu\nu}$ antisymmetric in indices⁷ give no contribution after integration in lk .

Integration limits in variable qk looks like

$$\begin{aligned}(qk)_- &= \sqrt{m_W^2 k^2} , \\ (qk)_+ &= \frac{1}{2} (m_W^2 + k^2) .\end{aligned}\tag{A.7}$$

As a result, we obtain the hadron multiplicity associated with the charm quark in a hadronic decay of the W boson (with the multiplicity of primary quark decay products subtracted):

$$\begin{aligned}\hat{N}_{cl} &= \frac{1}{(2\pi)^2 m_t^2 (1-r) H} \int_{Q_0^2}^{m_W^2} dk^2 \left[\frac{\partial}{\partial k^2} n_g(k^2) \right] \int_{(qk)_-}^{(qk)_+} d(qk) \\ &\quad \times \int_{(lk)_-}^{(lk)_+} d(lk) A(k^2, qk, lk) .\end{aligned}\tag{A.8}$$

The dimensionless quantity $n_g(k^2)$ in (A.8) is the multiplicity of hadrons in the gluon jet whose virtuality is k^2 , while the normalization H is given by

$$H = \frac{1}{6\pi^2} m_t^4 (1-r)^2 (1+2r) .\tag{A.9}$$

The analytic expressions for the functions $D_1(k^2, qk)$, $D_2(k^2, qk)$ in (A.2) are rather complicated to be shown here.⁸ That is why we present only the final results of our QCD calculations based on the formulae of this Appendix (see Eqs. (34), (35) in the main text).

Appendix B

In this Appendix we will demonstrate that the account of the W boson width results in only small corrections to the hadronic multiplicities.

⁷As the last term in Eq. (A.1).

⁸They depend also on the masses m_c and m_W .

The denominator of the W boson propagator (see the diagram in Fig. 9) is

$$q^2 - m_W^2 + im_W \Gamma_W , \quad (\text{B.1})$$

where m_W is the mass of the W boson, Γ_W its full width. The mean multiplicity is given by the formula

$$\langle n_h \rangle = \frac{1}{N} \int_0^{(m_t - m_b)^2} dq^2 \frac{m_W \Gamma_W}{(q^2 - m_W^2)^2 + m_W^2 \Gamma_W^2} F(q^2) n_h(q^2) , \quad (\text{B.2})$$

where n_h ($n_h = n_W$ or n_{tb} , see the main text) depends on the W boson virtuality q^2 . Here

$$F(q^2) = (m_t^2 - q^2)^2 (m_t^2 + 2q^2) , \quad (\text{B.3})$$

and the normalization N is

$$N = \int_0^\infty dq^2 \frac{m_W \Gamma_W}{(q^2 - m_W^2)^2 + m_W^2 \Gamma_W^2} F(q^2) . \quad (\text{B.4})$$

In both $\langle n_h \rangle$ (B.2) and N (B.3) the factor $m_W \Gamma_W$ is introduced, while common constants are omitted.

In the zero width limit,

$$\frac{m_W \Gamma_W}{(q^2 - m_W^2)^2 + m_W^2 \Gamma_W^2} \xrightarrow{\Gamma_W \rightarrow 0} \pi \delta(q^2 - m_W^2) , \quad (\text{B.5})$$

we do obtain that the mean multiplicity is equal to $n_h(m_W^2)$.

The numerical calculations with the use of formulae (B.2), (B.3) result in the following values:

$$\langle n_W \rangle = 19.04 , \quad (\text{B.6})$$

$$\langle n_{tb} \rangle = 16.37 . \quad (\text{B.7})$$

Thus, the account of non-zero width of the W boson slightly changes the average value of the multiplicities. Namely, n_W (54) has gone down by 0.3, while n_{tb} (55) has gone up by 0.23, but their sum remains almost unchanged.

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